Worksheet Four **MATH 104** Name:________________

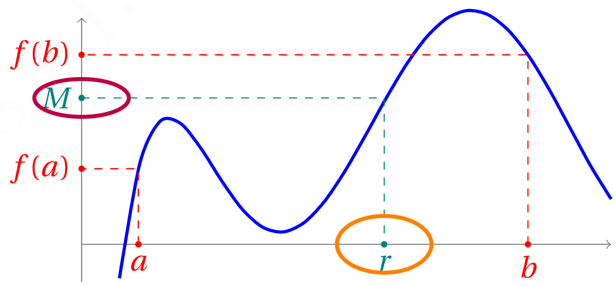
WORK ON THIS ASSIGNMENT IN GROUP OF 2-4. TURN IN YOUR WORK INDIVIDUALLY IN CLASS. YOU CAN USE YOUR NOTES FOR THIS ASSIGNMENT.

3.4: Graphs of Polynomials

A Few Definitions

- Global maximum: highest point on a graph; $f(a)$ where $f(a) \geq f(x)$ for all x.
- Global minimum: lowest point on a graph; $f(a)$ where $f(a) \leq f(x)$ for all x.
- **The Intermediate Value Theorem:**

  Let f be continuous on $[a, b]$ and M a value between $f(a)$ and $f(b)$. Then there exist number r in $[a, b]$ such that $f(r) = M$.



- Intermediate Value Theorem, in other words: for two numbers a and b in the domain of f, if $a < b$ and $f(a) \neq f(b)$, then the function f takes on every value between $f(a)$ and $f(b)$; specifically, when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis.
- **Existence of a Zero of a Continuous Function.**

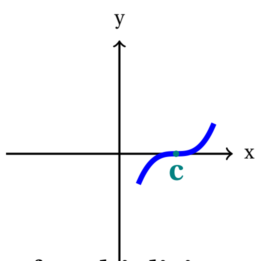
  Let f be continuous on $[a, b]$ and $f(a)$ and $f(b)$ have different signs, then f has a zero in $[a, b]$.
- **Multiplicity:** the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form $(x-h)^p$, $x = h$ is a zero of **multiplicity** p.
- There are no breaks or holes in the graph of a polynomial.

Sketching the Graph of a Polynomial Function.

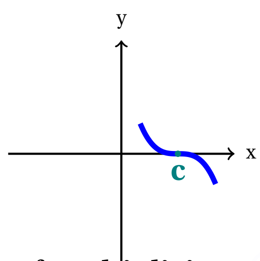
- Find the intercepts. x**-intercepts** are obtained by setting the $\underbrace{\text{function equal to zero}}_{f(x)=0}$. The y-**intercept** is obtained by $\underbrace{\text{plugging in zero}}_{f(0)}$ in the function.

- Check for **symmetry**. If the function is an even function, its graph is symmetrical about the y-axis, that is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x) = -f(x)$.

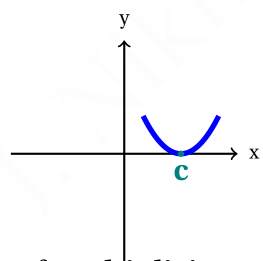
- Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.



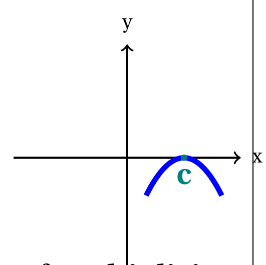
c : of **multiplicity** odd



c : of **multiplicity** odd

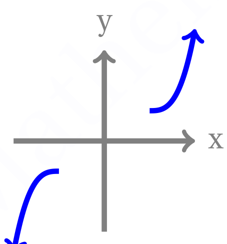


c : of **multiplicity** even

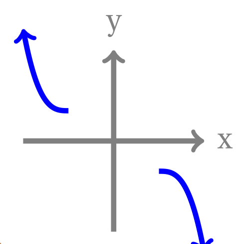


c : of **multiplicity** even

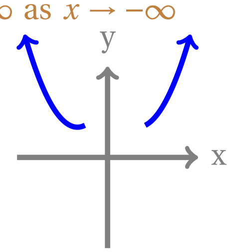
- Determine the end behavior by examining the leading term.



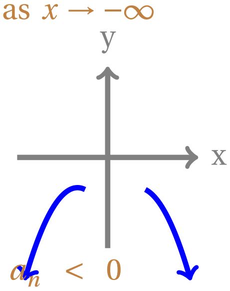
n: odd, $a_n > 0$
End behavior:
$y \to \infty$ as $x \to \infty$
$y \to -\infty$ as $x \to -\infty$



n: odd, $a_n < 0$
End behaviour:
$y \to -\infty$ as $x \to \infty$
$y \to \infty$ as $x \to -\infty$



n: even, $a_n > 0$
End behaviour:
$y \to \infty$ as $x \to \infty$
$y \to \infty$ as $x \to -\infty$



n: even, $a_n < 0$
End behaviour:
$y \to -\infty$ as $x \to \infty$
$y \to -\infty$ as $x \to -\infty$

- Use the **end behavior** and the **behavior at the intercepts** to sketch a graph.
- Use test points.
- Ensure that the number of turning points does **not exceed** one less than the degree of the polynomial.
- Optionally, use technology to check the graph.
- **An Interesting Result of Intermediate Value Theorem:** In solving inequalities involving polynomials only, we have been told to find the zeros of the polynomial and divide the number line into pieces and find the sign of each piece using test points. How we know that the sign doesn't change on each piece is using the Intermediate Value Theorem.

1. Which of the following describes the end behavior of $f(x) = -5x^4 + 3x^3 - 11x + 1$?

   (a) $y \to -\infty$ as $x \to \infty$ and $y \to \infty$ as $x \to -\infty$
   (b) $y \to -\infty$ as$x \to \infty$ and $y \to -\infty$ as $x \to -\infty$
   (c) $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$
   (d) $y \to \infty$ as $x \to \infty$ and $y \to \infty$ as $x \to -\infty$

2. Use the given information about the graph of the polynomial to express the polynomial as a product of factors (Find the rule of the polynomial function).

   (a) Degree = 3. Zeros at $x = -5$, $x = -2$,and $x = 1$. y-intercept at $(0, 6)$.
   (b) Degree = 4. Root of multiplicity 2 at $x = 4$, and a root of multiplicity 1 at $x = 1$ and $x = -2$. y-intercept at $(0, -3)$.
   (c) Degree = 3. With zeros -2, 0 and 5.

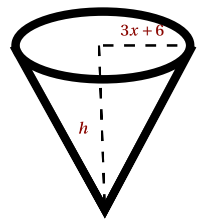
3. Find a polynomial with integer coefficients of degree 5 with zeros $1/3$, -2 and 5 of multiplicities 1,1 and 3 respectively.

4. Graph $P(x) = (x-1)(x-3)(x+1)$.

5. Graph $P(x) = -(x-1)(x-3)(x+1)+2$

6. Graph $p(x) = (x-2)^3(x+1)^2$.

7. A right circular cone has a radius of $3x+6$ units and a height 3 units less the radius.



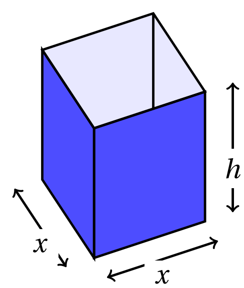
(a) Express the volume of the cone as a polynomial function. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$ for radius r and height h.

(b) What is the leading coefficient of this polynomial?

8. We are building a rectangular box with **no lid** with height h units and a square base with dimension x units. The box is made out of cardboard with total area of 48 unit^{2}s.

(a) Express h as a function of x.

Optional: Use the applet provided on this week's folder to graph the polynomial and find the maximum volume that can be attained.



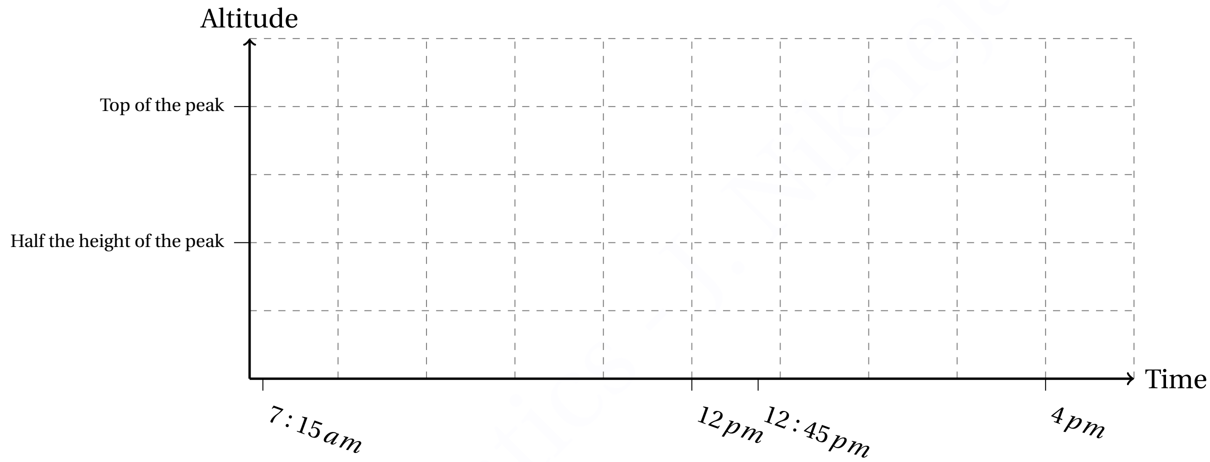
(b) Express the volume as a function of x, denoted by $V(x)$.

(c) What is the degree of polynomial $V(x)$?

The following question is from Math 125, Intermediate Value Section. Feel free to work on it or not. It is an optional work but it can be a fun puzzle.

9. **Learning Goal: Intermediate Value Theorem, The Runner Paradox.** Joe Started to climb Mount Damavand on Friday at 7 : 15 am and arrives on top of the peak at 4 pm on Friday. They started to climb down on Saturday at 7 : 15 am and arrived at the bottom at 4 pm on Saturday.

   (a) Let $F(t)$ be Joe's altitude on Friday as a function of time, t. Graph $F(t)$ if they take a 45 minutes break at Noon, they travel at a constant rate before and after the break, and they have traveled half the altitude of the peak by noon.

   (b) Explain in a sentence; why is the function $F(t)$ continuous?

   (c) Let $S(t)$ be the altitude of Joe on Saturday as a function of time, is $S(t)$ continuous?

   (d) Is $H(t) = S(t) - F(t)$ continuous?

   (e) Use the intermediate theorem to show that there was an altitude where Joe was at the same time on Friday and Saturday. Remember that you will need a function and an interval of time; the function values at the end points should have different signs.

Fun project: For the following polynomial, use a graphing calculator or this applet:

https://ggbm.at/uxvrnkzn to approximate local minima and maxima or the global minimum and maximum.

(a) $f(x) = x^3 - x - 1$

(b) $f(x) = x^4 - x^3 + 1$

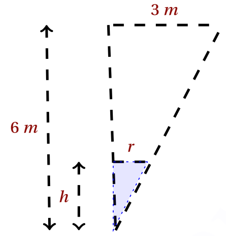
INDIVIDUAL WORK

UPLOAD TO CANVAS OR SUBMIT IN CLASS BEFORE DUE DATE. DISCUSSING THESE QUESTIONS IN YOUR GROUP IS ENCOURAGED BUT MAKE SURE YOU ARE TURNING IN YOUR OWN WORK.

10. **General Engineering:** In industry, liquids are kept in different shaped reservoirs; access to these is involved with the rate of change. This problem is shadowing this industrial concept. Please note that the **height** and **radius** of the water are related. Increase or decrease of one causes increase or decrease in the other respectfully. To find how they are related, use properties of similar triangles.

Water is being pumped into a tank shaped like a cone. The tank has height 6 meters and the diameter at the top is 6 meters.

(A) *(1 point)* Use the similarity of the two triangles to relate r (the radius of top of water in meters), h (the height of water in meters); denote this linear function by $r(h)$.

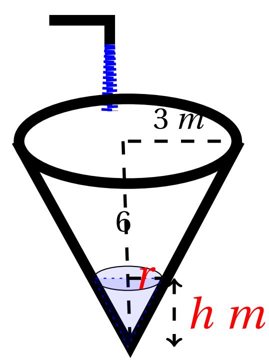


(B) *(1 point)* Express the volume of the water in the tank as a function of the height of water, h, in meters; denote this polynomial by $V(h)$.

(C) *(0.5 points)* What is the degree of the polynomial in Part (B), $V(h)$?

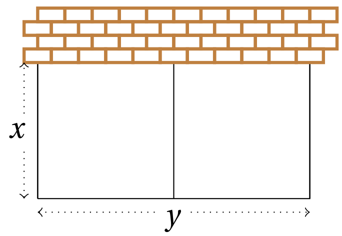
(D) *(0.5 points)* What is the leading coefficient of the polynomial in Part (B), $V(h)$?

(E) *(1 point)* What is the **average rate of change of the volume** when the height of the water increases from $h_1 = 2\ m$ to $h_2 = 4\ m$?



$$V = \frac{\pi r^2 h}{3}$$

11. Enrique is constructing a garden, against a wall, which will be separated into 2 plots as shown, where x and y are the width and length of the garden in yards:



He will surround three free sides of the garden by a fence, and separate the plots with fencing material. He have 60 yards of fencing material to use.

(A) *(0.75 points)* Express the dimension y as a function of x.

(B) *(0.75 points)* Find a function that models the **area** of the garden as a quadratic function of x; denoted by $A(x)$.

(C) *(1 point)* **An Optimization Problem:** What are the dimensions, x and y, that will maximize the area of the garden?

(D) *(0.5 points)* What is the maximum area?

12. *(1 point)* Perform the following operations and express the result as a simplified (in standard form $a+bi$) **complex number**. (Two out of four parts will be graded randomly.)

(a) $(8+4i)+(6-8i)=$

(b) $(6+8i)-(1-i)=$

(c) $(8+4i)\cdot(6+8i)=$

(d) $\dfrac{8+4i}{3+4i}=$